Section 6.8

Unilateral Laplace Transform

- The unilateral Laplace transform of the signal x, denoted $\mathcal{UL}\{x\}$ or X, is defined as

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt.$$

- The unilateral Laplace transform is related to the bilateral Laplace transform as follows:

$$\mathcal{UL}\{x\}(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t) u(t) e^{-st} dt = \mathcal{L}\{xu\}(s).$$

- In other words, the unilateral Laplace transform of the signal x is simply the bilateral Laplace transform of the signal xu.
- Since $\mathcal{UL}\{x\} = \mathcal{L}\{xu\}$ and xu is always a *right-sided* signal, the ROC associated with $\mathcal{UL}\{x\}$ is always a *right-half plane*.
- For this reason, we often *do not explicitly indicate the ROC* when working with the unilateral Laplace transform.

- With the unilateral Laplace transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral Laplace transform is *only invertible for causal signals*. In particular, we have

$$
\begin{aligned}
\mathcal{UL}^{-1}\{\mathcal{UL}\{x\}\}(t) &= \mathcal{UL}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\
&= x(t)u(t) \\
&= \begin{cases} x(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}
\end{aligned}
$$

- For a noncausal signal x, we can only recover x for $t > 0$.

- Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Frequency-Domain Scaling	$x(at),\ a > 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution Time-	$x_1 * x_2(t)$, x_1 and x_2 are causal	$X_1(s)X_2(s)$
Domain Differentiation Laplace-	$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$
Domain Differentiation Time-	$-tx(t)$	$\frac{d}{ds}X(s)$
Domain Integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Pair	$x(t),\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Solving Differential Equations Using the Unilateral Laplace Transform

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations.
- One common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions.

Part 7

Discrete-Time (DT) Signals and Systems

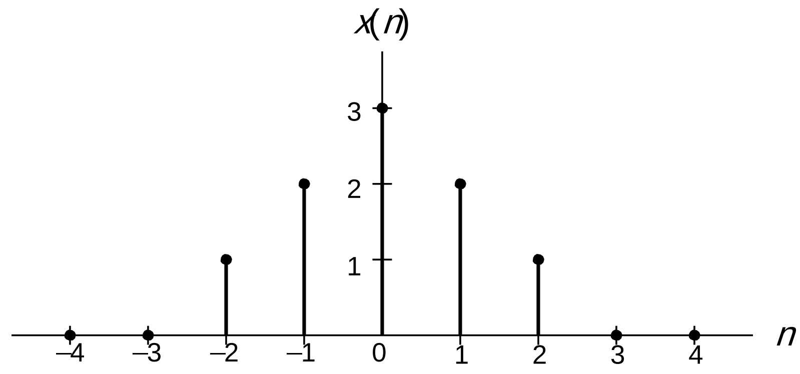
Section 7.1

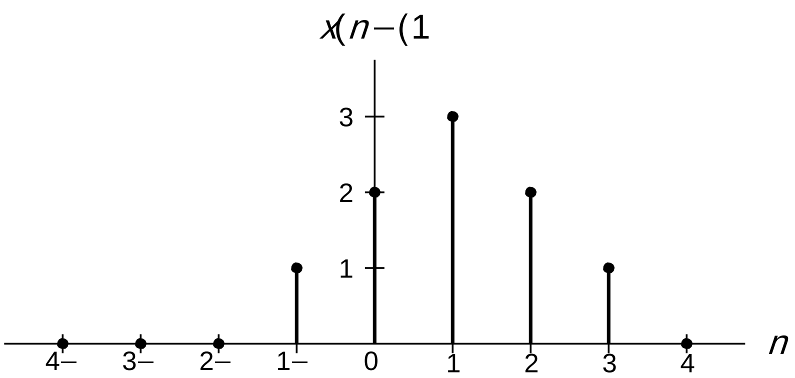
Independent- and Dependent-Variable Transformations

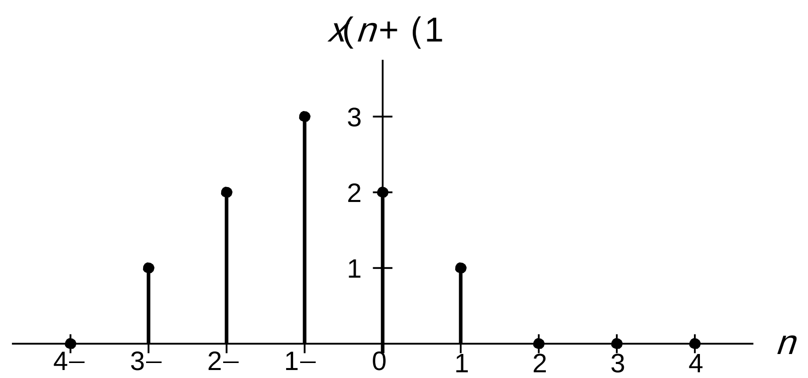
- Time shifting (also called translation) maps the input signal x to the output signal y as given by

$$y(n) = x(n-b),$$

where b is an integer.
- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $b > 0$, y is *shifted to the right* by $|b|$, relative to x (i.e., delayed in time). If
- $b < 0$, y is *shifted to the left* by $|b|$, relative to x (i.e., advanced in time).


x(n)
3
2
1
n
–4 –3 –2 –1 0 1 2 3 4

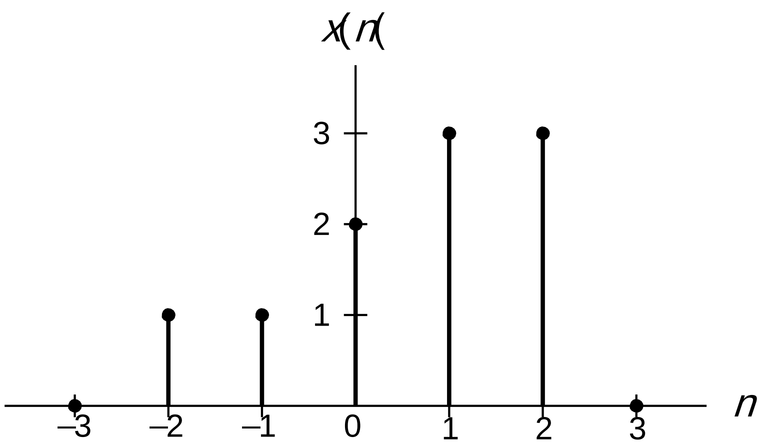

x(n−(1
3
2
1
n
4– 3– 2– 1– 0 1 2 3 4

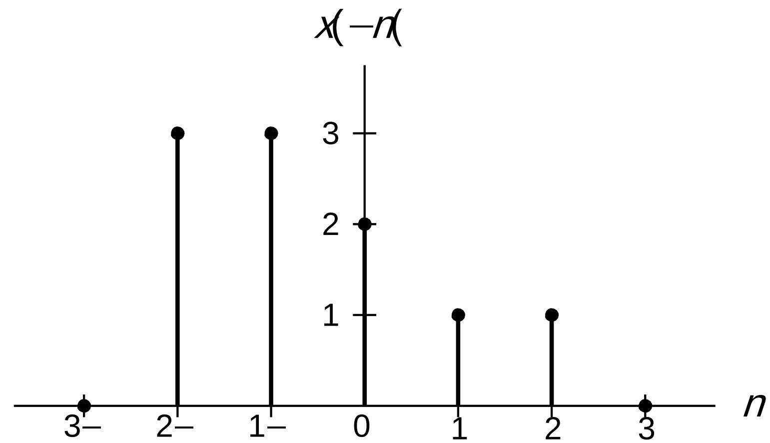

x(n+ (1
3
2
1
n
4– 3– 2– 1– 0 1 2 3 4

- Time reversal (also known as reflection) maps the input signal x to the output signal y as given by

$$y(n) = x(-n).$$

- Geometrically, the output signal y is a reflection of the input signal x about the (vertical) line $n = 0$.



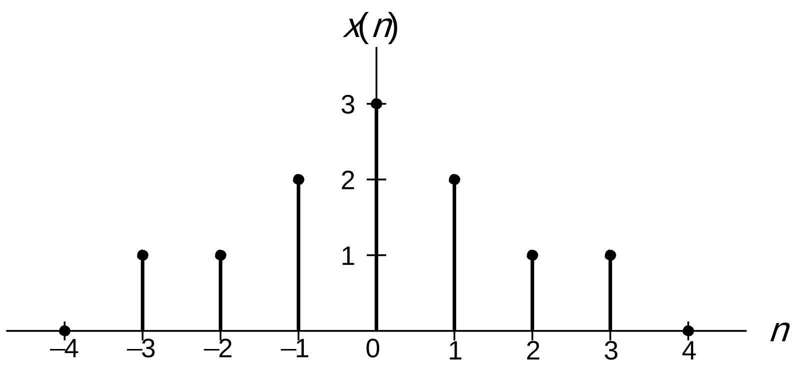


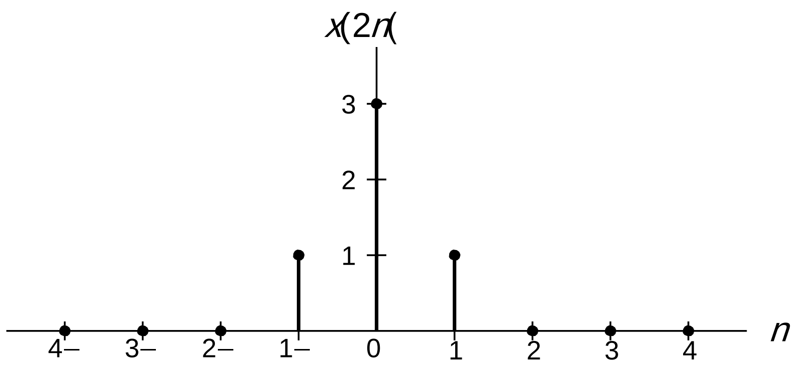
- Downsampling maps the input signal x to the output signal y as given by

$$y(n) = x(an)$$

where a is a *strictly positive* integer.

- The output signal y is produced from the input signal x by keeping only every ath sample of x.





- Consider a transformation that maps the input signal x to the output signal y as given by

$$y(n) = x(an - b),$$

where a and b are integers and $a \neq 0$.
- Such a transformation is a combination of time shifting, downsampling, and time reversal operations.
- Time reversal *commutes* with downsampling.
- Time shifting *does not commute* with time reversal or downsampling.
- The above transformation is equivalent to:
  1. first, time shifting x by b;
  2. then, downsampling the result by $|a|$ and, if $a < 0$, time reversing as well.
- If $\frac{b}{a}$ is an integer, the above transformation is also equivalent to:
  1. first, downsampling x by $|a|$ and, if $a < 0$, time reversing;
  2. then, time shifting the result by $\frac{b}{a}$.
- Note that the time shift is not by the same amount in both cases.

Section 7.2

Properties of Signals